Mathematical induction

Mathematical induction is a method of mathematical proof typically used to establish that a given statement is true for all natural numbers**.**

The simplest and most common form of mathematical induction proves that a statement involving a natural number *n* holds for all values of *n*.

The principle of mathematical induction is as follows:

For the statement $T(n)$, $n \in N$ is:

1) $T(1)$ **is true [or in some cases** $T(0)$ **is true]**

2) $T(n) \Rightarrow T(n+1)$ **is true for** $\forall n = 1,2,...$ **then, the statement** $T(n)$ **is true for** $\forall n \in N$

Practically, we will do:

1) Check whether the formula is correct for n = 1

2) Suppose that the formula is correct for n = k **[induction hypothesis]**

3) Argues that the formula is correct for n = k + 1

EXAMPLES:

1) Prove that the statement: $1+2+3+...+n = \frac{n(n+1)}{2}$ **holds for all natural numbers *n*.**

Proof:

i) Check whether the formula is correct for n = 1 (instead n , we put 1)

$$1 = \frac{1(1+1)}{2} \Rightarrow 1 = 1 \quad \text{correct}$$

ii) Induction hypothesis: suppose that the formula is correct for n = k (instead n, we put k)

$$1+2+3+...+k = \frac{k(k+1)}{2}$$

iii) Argues that the formula is correct for n = k + 1

First, see what you need to prove, in the initial formula replace n with k +1, but **always on the left side write before- last member.**

$$1+2+...+\underset{\uparrow}{k}+(k+1)=\frac{(k+1)(k+1+1)}{2} \quad \text{or} \quad 1+2+...+k+(k+1)=\frac{(k+1)(k+2)}{2}$$

before- last member **So, this should prove!**

Always **begin by hypothesis** that we have presumed that is always correct!

$$1+2+3...+k=\frac{k(k+1)}{2}$$

Stop a little, and compare the left side of the hypothesis and what should prove

$$1+2+3...+k=\frac{k(k+1)}{2}$$

$$1+2+...+k+\underset{\uparrow}{(k+1)}=\frac{(k+1)(k+2)}{2}$$

We see that in the hypothesis is missing member (k+1). It is a **trick,** that on both sides of the hypothesis add expression (k +1).

$$1+2+3...+k=\frac{k(k+1)}{2}$$

$$1+2+3...+k+(k+1)=\frac{k(k+1)}{2}+(k+1)$$

When you calculate, the right side must be $\frac{(k+1)(k+2)}{2}$

So: $$\frac{k(k+1)}{2}+\frac{k+1}{1}=\frac{(k+1)+2(k+1)}{2}=\frac{(k+1)(k+2)}{2}$$

This is a complete proof.

2) Prove that the statement: $1^2+2^2+3^2+...+n^2=\frac{n(n+1)(2n+1)}{6}$ **holds for all natural numbers *n*.**

Proof:

i) Check whether the formula is correct for n = 1

$$1^2=\frac{1(1+1)(2\cdot 1+1)}{6}\Rightarrow 1=1 \quad \text{correct}$$

ii) **Induction hypothesis:** suppose that the formula is correct for n = k

$$1^2+2^2+...+k^2=\frac{k(k+1)(2k+1)}{6}$$

iii) Argues that the formula is correct for n = k + 1

Always first we see what we have to prove:

$$1^2+2^2+...+k^2+(k+1)^2=\frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

or $$1^2+2^2+...+k^2+(k+1)^2=\frac{(k+1)(k+2)(2k+3)}{6}$$

Head from hypothesis, and on both sides add $(k+1)^2$

$$1^2+2^2+...+k^2=\frac{k(k+1)(2k+1)}{6}$$

$$\underbrace{1^2+2^2+...+k^2+(k+1)^2}=\underbrace{\frac{k(k+1)(2k+1)}{6}+(k+1)^2}$$

left side is same   This must give : $\frac{(k+1)(k+2)(2k+3)}{6}$

So: $$\frac{k(k+1)(2k+1)}{6}+\frac{(k+1)^2}{1}=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$=\frac{(k+1)\left[k(2k+1)+6(k+1)\right]}{6}$$

$$=\frac{(k+1)\left[2k^2+k+6k+6\right]}{6}$$

$$=\frac{(k+1)\left[2k^2+7k+6\right]}{6}$$

Term $2k^2+7k+6$,we'll separate on facts with $ak^2+bk+c=0$............................$a(k-k_1)(k-k_2)=0$

$$2k^2+7k+6=0$$

$$k_{1,2}=\frac{-7\pm 1}{4}$$

So: $2k^2+7k+6=2\left(k+\frac{3}{2}\right)(k+2)=(2k+3)(k+2)$

$$k_1=-\frac{3}{2}$$

$$k_2=-2$$

Let's go back to the task:

$$\frac{k(k+1)(2k+1)}{6}+\frac{(k+1)^2}{1}=\frac{k(k+1)(2k+1)+6(k+1)^2}{6}$$

$$=\frac{(k+1)[k(2k+1)+6(k+1)]}{6}$$

$$=\frac{(k+1)[2k^2+k+6k+6]}{6}$$

$$=\frac{(k+1)[2k^2+7k+6]}{6}$$

$$=\frac{(k+1)(2k+3)(k+2)}{6}$$

And this we should prove!

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3) Prove that the statement: $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$ **holds for all natural numbers *n*.**

Proof:

i) $\frac{1}{(2\cdot 1-1)(2\cdot 1+1)}=\frac{1}{2\cdot 1+1}\Rightarrow\frac{1}{3}=\frac{1}{3}$ correct

ii) Induction hypothesis: $\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$

iii) Argues that the formula is correct for n = k + 1

Prvo da vidimo šta treba da First to see what should be prove!

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

Proof will as usual start from the hypothesis:

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+...+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

on both sides will add $\frac{1}{(2k+1)(2k+3)}$

$$\frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+...+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}=\frac{k(2k+3)+1}{(2k+1)(2k+3)}=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$2k^2+3k+1=0 \qquad 2k^2+3k+1=a(k-k_1)(k-k_2)$$

$$k_{1,2}=\frac{-3\pm 1}{4} \qquad =2\left(k+\frac{1}{2}\right)(k+1)$$

$$k_1=-\frac{1}{2} \qquad =(2k+1)(k+1)$$

$$k_2=-1$$

Let's go back to the task:

$$\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}=\frac{k(2k+3)+1}{(2k+1)(2k+3)}=\frac{2k^2+3k+1}{(2k+1)(2k+3)}=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

***4)* Prove that $5^{n-1} + 2^n$ is divisible by 3 for all natural numbers *n*.**

Proof:

i) for n = 1 $5^{n-1} + 2^n = 5^{1-1} + 2^1 = 5^o + 2 = 1 + 2 = 3$ correct, because 3:3 = 1

ii) Induction hypothesis : $5^{k-1} + 2^k$ can be divided by 3

iii) To prove that $5^{n-1} + 2^n$ can be divided with 3 for n = k + 1

$$5^{n-1} + 2^n = 5^{k-1} + 2^{k+1}$$
$$= 5^{k-1+1} + 2^k \cdot 2^1$$
$$= 5^{k-1}5^1 \cdot + 2^k \cdot 2$$
$$= 5 \cdot 5^{k-1} + 2 \cdot 2^k$$

We use formula: $a^{m+n} = a^m \cdot a^n$

Write as"trick" : $5 \cdot 5^{k-1} = 3 \cdot 5^{k-1} + 2 \cdot 5^{k-1}$

$$= 5 \cdot 5^{k-1} + 2 \cdot 2^k$$
$$= 3 \cdot 5^{k-1} + 2 \cdot 5^{k-1} + 2 \cdot 2^k = 3 \cdot 5^{k-1} + 2(5^{k-1} + 2^k)$$

It is certainly divisible with 3 . Why?

$$3 \cdot 5^{k-1} + 2(5^{k-1} + 2^k)$$

obviously, because of 3 — is divisible by 3 because of our assumption that $5^{k-1} + 2^k$ is divisible by 3

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5) Prove that $6^{2n}+3^{n+2}+3^{n}$ is divisible by 11 for all natural numbers *n*.

Proof:

i) for n=1 is $6^{2n}+3^{n+2}+3^{n}=6^{2}+3^{3}+3^{1}$

$$=36+27+3$$

$$=66=6\cdot 11$$

correct

ii) Induction hypothesis $6^{2n}+3^{n+2}+3^{n}$ can be divided by 11

iii) Proof for n=k+1

$$6^{2(k+1)}+3^{k+1+2}+3^{k+1}=$$

$$6^{2k+1}+3^{k+2+1}+3^{k+1}= \qquad (a^{m+n}=a^{m}\cdot a^{n})$$

$$6^{2k}\cdot 6^{2}+3^{k+2}\cdot 3^{1}+3^{k}\cdot 3^{1}=$$

$$36\cdot 6^{2k}+3\cdot 3^{k+2}+3\cdot 3^{k}=$$

Now, we need some ideas!

As with 6^{2k} we have 36, triples with 3^{k+2} and 3^{k} we will write as 3 = 36 - 33

$36\cdot 6^{2k}+\underset{36-33}{\boxed{3}}\cdot 3^{k+2}+\underset{36-33}{\boxed{3}}\cdot 3^{k}$ is idea!

So:

$$36\cdot 6^{2k}+36\cdot 3^{k+2}-33\cdot 3^{k+2}+36\cdot 3^{k}-33\cdot 3^{k}=$$

$$=36(6^{2k}+3^{k+2}+3^{k})-33(3^{k+2}+3^{k})$$

Term $36(6^{2k}+3^{k+2}+3^{k})$ is divisible by 11 because if assumptions and expression $33(3^{k+2}+3^{k})$ because of number 33 = 3 *11

This is a complete proof.

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6) Prove that for any natural number $n>1$ is $\frac{1}{n+1}+\frac{1}{n+2}+...+\frac{1}{2n}>\frac{13}{24}$

Solution:

i) For n=2 is $\frac{1}{3}+\frac{1}{4}=\frac{7}{12}=\frac{14}{24}$

$\frac{14}{24}>\frac{13}{24}$ correct

ii) Induction hypothesis: for n=k $\frac{1}{k+1}+\frac{1}{k+2}+...+\frac{1}{2k}>\frac{13}{24}$

iii) For n= k+ 1, we need to prove: $\frac{1}{k+2}+\frac{1}{k+3}+...+\frac{1}{2k}+\frac{1}{2(k+1)}>\frac{13}{24}$

We must use a new "trick"!

Mark with:

$$S_k=\frac{1}{k+1}+\frac{1}{k+2}+...+\frac{1}{2k} \quad (S_k>\frac{13}{24}\text{ ,By assumption)}$$

and $$S_{k+1}=\frac{1}{k+2}+\frac{1}{k+3}+...+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2(k+1)}$$

Determine the difference $S_{k+1}-S_k$ =?

$$S_{k+1}-S_k=\left(\frac{1}{k+2}+\frac{1}{k+3}+...+\frac{1}{2k+1}+\frac{1}{2(k+1)}\right)-\left(\frac{1}{k+1}+\frac{1}{k+2}+...+\frac{1}{2k}\right)$$

$$=\frac{1}{k+2}+\frac{1}{k+3}+...+\frac{1}{2k}+\frac{1}{2k+1}+\frac{1}{2(k+1)}-\frac{1}{k+1}-\frac{1}{k+2}-...-\frac{1}{2k}$$

$$=\frac{1}{2k+1}+\frac{1}{2(k+1)}-\frac{1}{k+1}$$

$$=\frac{1\cdot 2(k+1)+1\cdot(2k+1)-2(k+1)}{(2k+1)\cdot 2\cdot(k+1)}$$

$$=\frac{2k+2+2k+1-4k-2}{2(2k+1)(k+1)}$$

$$=\frac{1}{2(2k+1)(k+1)}>0$$

This is certainly positive, because $k > 0 \quad 2k+1 > 0$ and $k+1 > 0$

So: $S_{k+1} - S_k > 0$

$$S_{k+1} > \underbrace{S_k > \frac{13}{24}}_{\text{induction hypothesis}}$$

Then is: $S_{k+1} > \frac{13}{24}$

This is a complete proof !

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7) Prove that: $2^n > n^2$ **for everyone** $n \geq 5$

Proof:

Proof begin with $n = 5$, because $n \geq 5$

i)

$$n = 5 \Rightarrow 2^5 > 5^2$$
$$36 > 25 \text{ correct}$$

ii) Induction hypothesis: for n=k is $2^k > k^2$

iii) For n= k+ 1, we need to prove: $2^{k+1} > (k+1)^2$

Here we need new ideas!

Watch the term : $\left(1+\frac{1}{n}\right)^2$

$$\left(1+\frac{1}{n}\right)^2 = 1^2 + 2\cdot 1\cdot \frac{1}{n} + \left(\frac{1}{n}\right)^2 = 1 + \frac{2}{n} + \frac{1}{n^2}$$

since $n \geq 5 \Rightarrow \frac{1}{n} \leq \frac{1}{5}$ and $\frac{1}{n^2} \leq \frac{1}{25}$

then $\left(1+\frac{1}{n}\right)^2 = 1 + \frac{2}{n} + \frac{1}{n^2} \leq 1 + \frac{2}{5} + \frac{1}{25}$

$$1+\frac{2}{5}+\frac{1}{25}=1+\frac{11}{25}<2$$

$$\left(1+\frac{1}{n}\right)^2=\left(\frac{n+1}{n}\right)^2<2$$

$$\frac{(n+1)^2}{n^2}<2$$

and then is $\frac{(k+1)^2}{k^2}<2$, and hypothesis is $2^k>k^2$

$$\left.\begin{array}{l} 2^k>k^2 \\ 2>\frac{(k+1)^2}{k^2} \end{array}\right\}$$ multiply them! (left with left and right with right)

$$2^k\cdot 2>\frac{(k+1)^2}{k^2}\cdot k^2$$

$2^{k+1}>(k+1)^2 \rightarrow$ **and this is what we want to prove!**